

# THE EFFECTS OF CHANGES IN MORTALITY AND FERTILITY ON AGE COMPOSITION

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THE age composition of any population is determined wholly by the past history of its births and deaths at each age, and by the number and age of the migrants who have entered or left the population. In the 19th century and the first half of the 20th, many national populations have been characterized by quite substantial changes in birth and death rates, and, inevitably, by considerable changes in age composition. Our purpose is to analyze as generally as we can the effects of changed vital rates on populations undisturbed by migration.

We cannot provide a usefully clear analysis of the general case of an arbitrary initial age distribution, arbitrary initial vital rates, and arbitrary vital rate changes. One reason that this case is too complicated is that the age distribution would typically change from its initial form even if the birth and death rates were to remain unaltered. In other words, most age distributions have unavoidable alterations built in—for example, a small cohort due to a pronounced birth deficit for a few years will remain smaller than neighboring cohorts as it becomes older and there will be a hollow moving out through

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This paper is an extension of earlier work on the same problem. Much of the fundamental analysis is found in Alfred J. Lotka's publications, summarized in his *THÉORIE ANALYTIQUE DES ASSOCIATIONS BIOLOGIQUES*, 1939. Three recent articles anticipate our principal results, though they argue mostly by example rather than analytically. These articles are by Frank Lorimer, 1951; Alfred Sauvy, 1954; and the Population Division of the United Nations, 1954.

the age distribution with the passage of time. In order to isolate the effects of vital rate changes from these built-in changes, we will assume, for the most part in what follows, that the initial population has the so-called stable age distribution—a construct of population analysis partly originated and thoroughly explored by Lotka [Lotka, 1939]. The stable age distribution is the age distribution which would ultimately be established in a closed population if it maintained fixed schedules of fertility and mortality—the only age distribution with no built-in change.

Vital rate changes will be analyzed in two different ways. First, we will consider the ultimate age distribution which would arise from the indefinite continuation of new vital rates. This distribution is independent of the time sequence by which the rates are introduced. The first form of analysis will contrast, in other words, the new stable age distribution implicit in the new schedule of rates with the old distribution implicit in the old schedule. The second kind of analysis will explore the immediate effects of vital rate changes occurring in a particular time sequence. The ultimate stable age distribution is analogous to the “steady-state” response of a mechanical or electrical system to a new set of forces, while the changes in response to a particular sequence of rates resemble the transient characteristics of a physical system. The analogy suggests (correctly) that the steady-state response presents the easier problems to solve.

Throughout our discussion, the population considered will consist entirely of females: only female births, female fertility rates, and female deaths are treated. *Mutatis mutandis*, the same analysis would apply with equal validity to males. One needs to analyze the two sexes simultaneously only out of a concern for the logical consistency of the assumed vital rate changes for each sex. We will take advantage of the relative simplicity of analyzing one sex, and will consider as secondary such questions as the effect of the availability of spouses in determining fertility.

DIFFERENCES BETWEEN STABLE AGE DISTRIBUTIONS ARISING FROM DIFFERENT MORTALITY AND FERTILITY RATES

If the proportion of females born alive who survive to age  $a$  is  $l_a$ , and the probability of bearing a female child at age  $a$  is  $m(a)$ , and if these functions remain unchanged, a constant rate of growth will eventually be established. The age distribution will become:

$$(1) \quad c(a) = \frac{n(a)}{N} = \frac{e^{-ra}l_a}{\int_0^{\omega} e^{-ra}l_a da} = be^{-ra}l_a$$

where  $c(a)$  is the proportion of the female population at age  $a$ ,  $n(a)$  is the number at age  $a$ ,  $N$  is the total number of females,  $\omega$  is the oldest age attained,  $r$  is the constant rate of growth finally established, and  $b$  is the female birth rate when growth rate has become  $r$  [Lotka, 1939].

The effect of the fertility schedule,  $m(a)$ , in this scheme, can be seen when one considers that the product of the number of women at each age by the fertility schedule summed for all ages must equal the number of births.

$$\begin{aligned} \int_0^{\omega} c(a)m(a)da &= b, \text{ or} \\ \int_0^{\omega} be^{-ra}l_a m(a)da &= b, \text{ or} \\ (2) \quad \int_0^{\omega} e^{-ra}l_a m(a)da &= 1 \end{aligned}$$

Our problem is to compare two age distributions,  $c(a)$  and  $c'(a)$ , given two mortality schedules,  $l_a$  and  $l'_a$ , and two fertility schedules,  $m(a)$  and  $m'(a)$ . The two may be compared by taking the ratio of one age distribution to the other:

$$(3) \quad \frac{c'}{c}(a) = \frac{b'}{b} \frac{l'_a}{l_a} \frac{e^{-r'a}}{e^{-ra}} = \frac{b'}{b} \frac{l'_a}{l_a} e^{-\Delta ra}$$

where  $\Delta r$  is the difference between the two growth rates.<sup>2</sup>

We will observe at what ages this ratio  $\frac{c'}{c}$  increases, at what ages it decreases (and by how much) in response to differences in mortality and fertility.

*Stable Age Distributions with the Same Mortality but Different Fertility Schedules.* We first consider the ratio of two stable age distributions where  $l'_a = l_a$  at all ages, but  $m'(a) \neq m(a)$ . Under these circumstances equation (3) takes the simple form

$$(4) \quad \frac{c'}{c}(a) = \frac{b'}{b} e^{-\Delta r a}$$

The ratio equals  $\frac{b'}{b}$  at birth, and declines 100 $\Delta r$  per cent per year thereafter. At an age  $\hat{a}$  approximately equal to the average of the mean ages in the two stable age distributions,<sup>3</sup> the ratio passes through unity.

Thus the higher fertility age distribution always has a greater proportion at the young ages, and a smaller proportion at the

<sup>2</sup> We will always designate the age distribution with the larger growth rate by a prime, so that  $\Delta r$  will always be positive.

<sup>3</sup> Lotka has shown (p. 25) that  $\hat{a} = \lambda_1 - \frac{\lambda_2}{2}(r+r') + \frac{\lambda_3}{3!}(r^2+r'^2) + \dots$  where  $\lambda_n = n^{\text{th}}$  Thielian semi-invariant of the life table. But  $\frac{\bar{a} + \bar{a}'}{2} = \lambda_1 - \frac{\lambda_2}{2}(r+r') + \frac{\lambda_3}{4}(r^2+r'^2) + \dots$

Thus the difference between  $\hat{a}$  and  $\frac{\bar{a} + \bar{a}'}{2}$  is approximately  $\frac{\lambda_3}{12}(\Delta r)^2$ , which is usually negligible (for the largest observed value of  $\lambda_3$ , and  $\Delta r = .03$ , this difference is about one-third of a year).

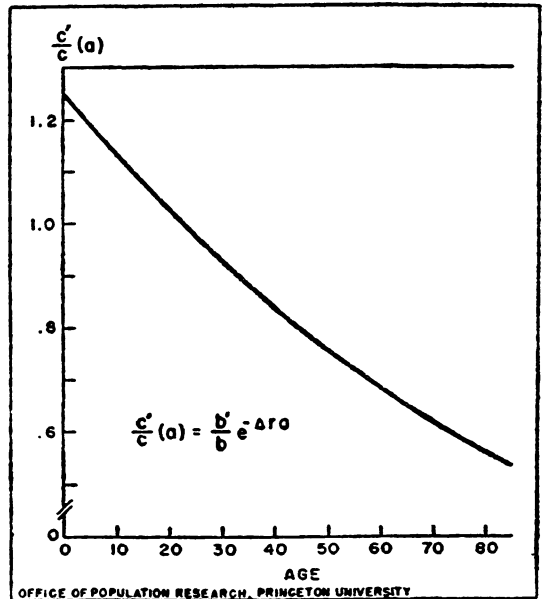


Figure 1.  $\frac{c'}{c}(a)$  for stable age distributions, with the same mortality but different fertility.

old ages, the dividing point being (loosely speaking) the mean age of the population. (See Figure 1.)

How much difference in fertility is required to produce a given ratio of birth rates and a given difference in  $r$ ? This question is readily answered when the two fertility schedules have the same form, with one being the same multiple of the other at all ages. Thus if  $m'(a) = K \cdot m(a)$ ,

$$(5) \quad \Delta r = \frac{\log_e K}{T} \text{ (approximately)}^4$$

where  $T$  is the mean length of generation.

From (4) we note that  $\frac{c'}{c} (\hat{a}) = \frac{b'}{b} e^{-\Delta r \hat{a}} = 1$ , or

$$(6) \quad \frac{b'}{b} = e^{\Delta r \hat{a}}$$

Combining (5) and (6) we can see that

$$(7) \quad \log_e \frac{b'}{b} \cong \frac{\hat{a}}{T} \log_e \frac{m'(a)}{m(a)}, \text{ and}$$

<sup>4</sup> This approximate expression can be justified as follows:

(a)  $e^{-rT} R_0 = e^{-r'T'} R'_0 = 1$  (where  $R_0$  and  $R'_0$  are net reproduction rates, and  $T$  is the mean length of generation);

(b) but  $R'_0 = \int_0^{\omega} m'(a) l_a da = K R_0$ , hence

(c)  $e^{r'T'} - rT = K$

but  $T = \frac{R_1}{R_0} + \frac{1}{2} \left\{ \left( \frac{R_1}{R_0} \right)^2 - \frac{R_2}{R_0} \right\} r + \dots$  (cf. Lotka, 1939, p. 69)

where  $R_n = \int_0^{\omega} a^n l_a m(a) da$

Note that  $R'_n = K R_n$ , hence

(d)  $T' \cong T + \frac{1}{2} \left\{ \left( \frac{R_1}{R_0} \right)^2 - \frac{R_2}{R_0} \right\} (r' - r)$ .

However,  $\frac{T}{2} < -\frac{1}{2} \left\{ \left( \frac{R_1}{R_0} \right)^2 - \frac{R_2}{R_0} \right\} \leq T$  empirically for a wide range of fertility and

mortality schedules, hence  $T(1 - \Delta r) \leq T' < T \left( 1 - \frac{\Delta r}{2} \right)$ .

(c) can be rewritten:  $r'T' - rT = \log K$ , and thus

(e)  $\frac{\log K}{T \left( 1 - \frac{r'}{2} \right)} < \Delta r \leq \frac{\log K}{T(1 - r)}$ ;

hence (5) can be off at most by a factor of  $\frac{1}{1 - r'}$ . However  $|r'| \leq .03$  in almost every instance; (5) therefore holds to within about 3 per cent.

$$(8) \quad \log_e \frac{c'}{c}(a) \cong \frac{\hat{a} - a}{T} \log_e \frac{m'(a)}{m(a)}$$

Whether the ratio of the birth rates is greater or less than the ratio of fertilities depends on whether  $\hat{a}$  is greater or less than  $T$ . This result is quite in accord with common sense, since all ages less than  $\hat{a}$  have proportionately greater numbers when fertility is high. When  $\hat{a}$  is greater than  $T$ , the central age of childbearing (to characterize  $T$  loosely) has relatively greater numbers in the high fertility age distribution. High fertility is thus combined with a favorable age distribution to produce a more than proportionate rise in the birth rate.

If the two fertility schedules do not differ merely in level, but also in form, equation (5) above will no longer hold. In the more general case,  $\Delta r$  may be expressed as follows:

$$(9) \quad \Delta r = \frac{\log_e \frac{R'_0}{R_0} - r\Delta T}{T'}$$

If the fertility schedules differed only in the age at which each level of fertility was reached, the principal determinant of  $\Delta r$  would be the difference in  $T$ . If  $m'(a) = m(a - x)$ ,  $\Delta T$  would be approximately equal to  $-x$  and  $\log \frac{R'_0}{R_0}$  would differ from 0 only because the  $m'$  schedule of fertility, occurring at younger ages, would be combined with lower mortality rates. In this instance, (9) can be written, approximately:

$$(10) \quad \Delta r \cong \frac{\log_e \frac{l_{T-x}}{l_T} + r x}{T - x} \cong \frac{{}_x q_{T-x} + r x}{T - x}$$

where  ${}_x q_{T-x}$  is the probability of dying between ages  $T - x$  and  $T$  (Dublin and Lotka, 1925).

Differences in fertility, in sum, produce a very simple difference in stable age distributions: one age distribution has a higher initial ordinate and a steeper slope than the other, larger fractions of its population in the younger ages, an equal fraction at the mean age, and smaller fractions at the older ages.

*Stable Age Distributions with the Same Fertility but Different Mortality Schedules.* The effect of different mortality rates on the stable age distribution is more complicated than the effect of differences in fertility.

The general expression for the ratio of the two age distributions is:

$$(3) \quad \frac{c'}{c} (a) = \frac{b'l'_a}{bl_a} e^{-\Delta ra}$$

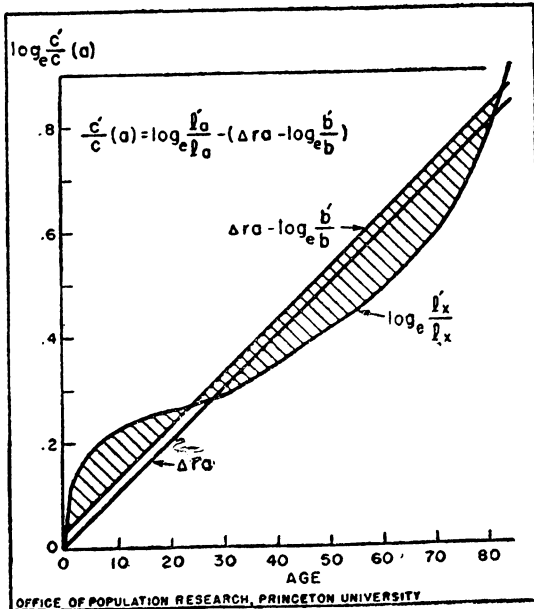
In its logarithmic form, this equation becomes

$$(11) \quad \log \frac{c'}{c} (a) = \log \frac{l'_a}{l_a} - \left( \Delta ra - \log \frac{b'}{b} \right)$$

Figure 2 represents the right-hand side of equation (11) geometrically, and isolates the factors which determine the differences in two stable age distributions.

The vertical dimension of the cross-hatched area represents  $\log \frac{c'}{c} (a)$ , being  $\log \frac{l'_a}{l_a} - \left( \Delta ra - \log \frac{b'}{b} \right)$ . Where the distance

Figure 2. The ratio of two stable age distributions, with the same fertility but different mortality.



between the two lines is great, the ratio of  $\frac{c'}{c} (a)$  differs substantially from unity. The two lines intersect at ages where the two distributions have the same proportions.

We will next describe the basis for the geometric constructions in Figure 2 which enable us to determine  $\Delta r$  and  $\frac{b'}{b}$  from a knowledge (primarily) of  $\log \frac{l'_a}{l_a}$ .

These geometric constructions are based on two equations, and an approximation to  $\log x$  when  $|x - 1|$  is small.

The first of the equations underlying our construction follows directly from (2):

$$(12) \quad \int_{a_1}^{a_2} (e^{-r'a}l'_a - e^{-ra}l_a)m(a) da = 0$$

where  $a_1$  and  $a_2$  are the youngest and oldest ages of childbearing.

The second equation results from the necessity for each age distribution to total 100 per cent:

$$(13) \quad \int_0^{\omega} (b'e^{-r'a}l'_a - be^{-ra}l_a) da = 0$$

We will now consider the approximation  $\log(x) \cong x - 1$  (when  $x$  is near to 1) as applied to  $\log\left(\frac{l'_a}{l_a} e^{-\Delta ra}\right)$ .

$$(14) \quad \log\left(\frac{l'_a e^{-r'a}}{l_a e^{-ra}}\right) \cong (e^{-r'a}l'_a - e^{-ra}l_a) \frac{1}{e^{-ra}l_a}$$

When (14) is compared with (12), it becomes clear that

$$\int_{a_1}^{a_2} \log\left(\frac{l'_a}{l_a} e^{-\Delta ra}\right) e^{-ra}l_a m(a) da \cong 0, \text{ or that}$$

$$(15) \quad \int_{a_1}^{a_2} \left( \log \frac{l'_a}{l_a} - \Delta ra \right) c(a)m(a) da \cong 0$$

Equation (15) tells us that from ages 15 to 45 the positive areas between  $\log \frac{l'_a}{l_a}$  and  $\Delta ra$  are approximately balanced by the negative areas when the areas are "weighted" by the number of births at each age of mother. This relation enables us to estimate  $\Delta r$  quite closely by graphical methods. One plots  $\log\left(\frac{l'_a}{l_a}\right)$ , draws a vertical line at 15 and one at 45, and pivots a straight line on the origin, until, making allowance for weighting, the positive and negative areas between the straight and curved lines are balanced in the interval 15 to 45.

In Figure 3 the straight line is adjusted until the two shaded areas, weighted by the curve drawn directly beneath, are equal.



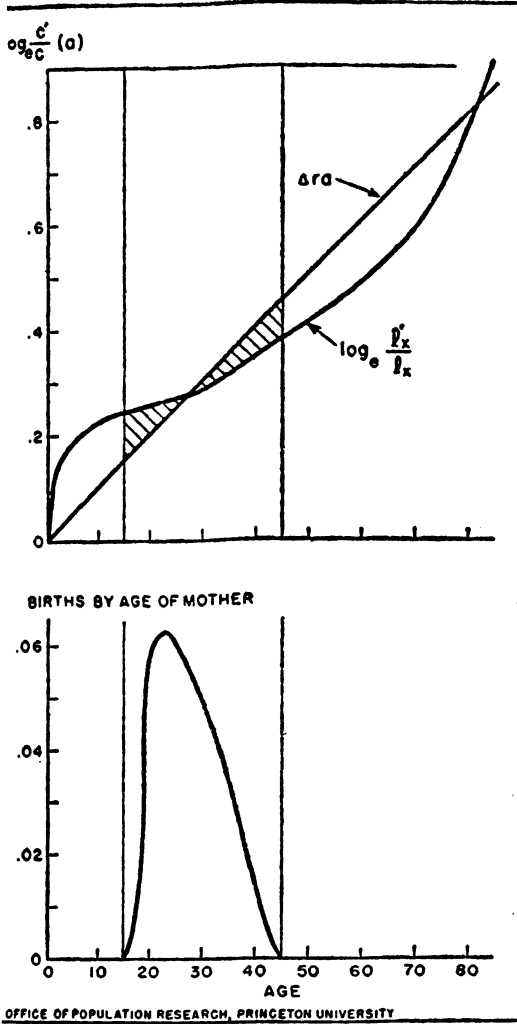


Figure 3. Determining the change in the intrinsic rate of growth, with changing mortality and constant fertility.

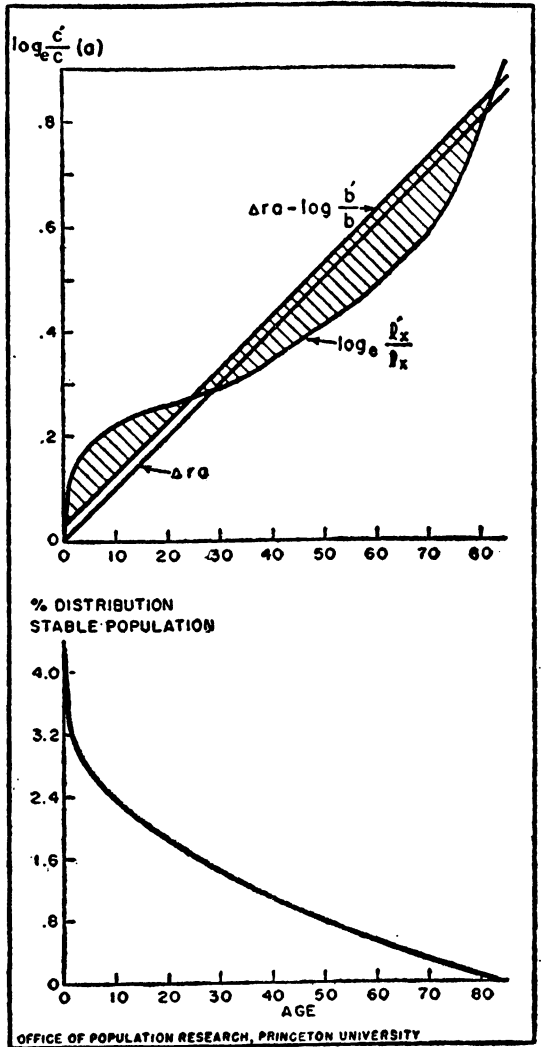


Figure 4. Determining the ratio of birth rates and the ratio of the proportions at each age in two stable age distributions, with different mortality but the same fertility.

There is a simple common sense basis for equalizing these areas. The distance between the two lines represents the log of the ratio of one age distribution to the other, when both are drawn from the same initial ordinate—when we assume  $b' = b$ . If the birth rates *were* equal, the excess mothers of one age distribution at some ages (weighted by their childbearing rates)

would have to be balanced by a relative deficit of mothers at other ages (weighted by *their* childbearing rates). In other words, if we equalize the births, the assumption of constant fertility requires that childbearing women should also be in some sense equalized.

There remains the determination of  $\frac{b'}{b}$ . By the same approximation used in (14), we may express  $\log \frac{c'}{c}(a)$  as:

$$(16) \quad \log \frac{c'}{c}(a) \cong \left( \frac{c'}{c}(a) - 1 \right) = (c'(a) - c(a)) \frac{1}{c(a)}$$

When (16) is compared with (13), it becomes clear that

$$(17) \quad \int_0^{\omega} \left( \log \frac{c'}{c}(a) \right) c(a) da \cong 0, \text{ or that}$$

$$\int_0^{\omega} \left\{ \log \frac{l'_a}{l_a} - \left( \Delta r a - \log \frac{b'}{b} \right) \right\} c(a) da \cong 0$$

$\log \frac{b'}{b}$  can be estimated by a graphic method similar to that employed for estimating  $\Delta r$ . One draws a straight line parallel to  $\Delta r a$  such that the positive areas between the line and  $\log \frac{l'_a}{l_a}$  balance the negative areas, when both areas are weighted by  $c(a)$ . In Figure 4, the positive shaded areas are made equal to the negative shaded areas, when the areas are weighted by the age distribution  $c(a)$ , sketched at the bottom of Figure 4.

The analysis of mortality differences might end here were it not for the fact that many actual mortality changes conform to a rather simple pattern, which can be expressed as the sum of three even simpler components. The total effect on the stable age distribution of a typical change from one life table to another is the same as if these three component changes had occurred sequentially.

The three components of the typical pattern found in life table changes are labeled A, B, and C in Figure 5.

Curve D represents the combination (sum) of A, B, and C. The typical pattern consists of a relatively sharp rise of  $\log \frac{l'_a}{l_a}$  beginning at age zero and extending (with diminishing slope)

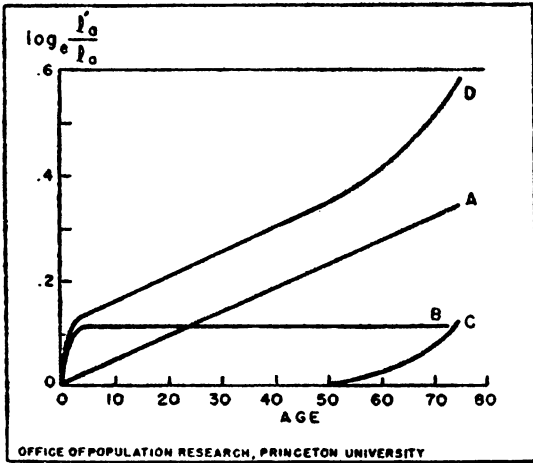


Figure 5. Components of  $\log_e \frac{l'_a}{l_a}$  in a typical change in mortality.

through the early childhood years, a nearly linear rise extending from age 5 to age 50, 60, or beyond, and section of increasing steepness in the older ages. The linear portion is extended and lowered so that it passes through the origin, forming A in Figure 5. The excess improvement in  $\log \frac{l'_a}{l_a}$  in childhood is represented by B, and the excess in the older ages by C. We will now show the effects of component A—a linear  $\log \frac{l'_a}{l_a}$  that passes through the origin.

(i) *Curve A in Figure 5.* The linear rise in  $\log \frac{l'_a}{l_a}$  is the simplest to analyze. Assume  $\log \frac{l'_a}{l_a} = sa$ , where  $s$  is constant. Under these circumstances,  $l'_a = l_a e^{sa}$ . Also  $\Delta r = s$ ,  $\frac{b'}{b} = 1$ , and the two age distributions are identical! This type of mortality difference, of course, is one where the slope of  $\log \frac{l'_a}{l_a}$  is constant. Now the slope of  $\log \frac{l'_a}{l_a}$ —which is equal to  $\frac{d}{da} (\log l'_a - \log l_a)$ —equals the difference in what is called the force of mortality at age  $a$ . If the slope is constant, the change in the force of mortality is

the same for all ages. A direct implication of the constant slope is that the probability of surviving for a given period (say a year) is changed in the same proportion at all ages. If the proportionate change in the probability of surviving at every age is the same, the two stable populations will have growth rates differing by an amount equal to the uniform difference in the probability of surviving, but will have precisely the same age distributions.<sup>5</sup> This result can be readily understood by observing that the higher probability of surviving (which with a given growth rate would tend to make the population older) is exactly offset by the higher growth rate (which with a given life table would tend to make the population younger).

(ii) *Curve B in Figure 5.* The B curve in Figure 5 represents an improvement in mortality concentrated in the early childhood ages. Such a difference has an effect similar to that of different fertility. In fact, if the mortality difference were wholly in the first few moments after birth (to take the extreme case), it would be indistinguishable from a difference in fertility. The age distribution is indifferent, we might say, as to whether a greater flow of infants emerges from a higher birth rate or from a lower infant mortality rate.

Assume, then, that  $\frac{l'_a}{l_a} = k$  for  $a \geq \epsilon$ ,  $\epsilon$  being a small positive

<sup>5</sup> This conclusion is seen immediately from the graphical technique outlined above for determining  $\Delta r$  and  $\log \frac{b'}{b}$ . It can also be shown analytically as follows:

Let  $l'_a = l_a e^{sa}$

Then from (12):

$$\int_{a_1}^{a_2} (e^{-r'a} l'_a e^{sa} - e^{-ra} l_a) m(a) da = 0$$

or

$$\int_{a_1}^{a_2} (e^{-(r'-s)a} - e^{-ra}) l_a m(a) da = 0$$

With  $l_a$  and  $m(a)$  everywhere non-negative, this last equation can hold only if  $r' = r + s$ , or if  $\Delta r = s$ .

But  $c(a) = \frac{e^{-ra} l_a}{\int_0^{\omega} e^{-ra} l_a da}$ ,

and  $c'(a) = \frac{e^{-r'a} l'_a}{\int_0^{\omega} e^{-r'a} l'_a da} = \frac{e^{-(r+s)a} e^{sa} l_a}{\int_0^{\omega} e^{-(r+s)a} e^{sa} l_a da} = c(a)$

number. Consider the substitution of  $l'_a$  for  $l_a$  in equation (2). We obtain

$$(18) \quad \int_{a_1}^{a_2} e^{-r'a} l'_a m(a) da = k \int_{a_1}^{a_2} e^{-r'a} l_a m(a) da = 1$$

This is precisely the same expression one would obtain if the life tables were the same, and  $m'(a) = km(a)$ . Hence the difference in  $r$  is precisely the same whether one considers a different life table with  $k$  times the probability of surviving to an early age and unchanged probabilities from that age on, or considers a different fertility schedule with  $k$  times the fertility level at all ages.

Thus for this assumed difference in mortality,

$$(19) \quad \Delta r \cong \frac{\log \frac{l'_\epsilon}{l_\epsilon}}{T} \quad (\text{from considerations similar to those underlying (5)})$$

and

$$(20) \quad \log \frac{c'}{c}(a) \cong \frac{\hat{a} - a}{T} \log \frac{l'_\epsilon}{l_\epsilon}$$

(from considerations similar to those underlying (8))

However, the ratio of the intrinsic birth rates would be quite different when the difference in age distribution arises from a difference in infant mortality rather than a difference in fertility.

Since  $\frac{c'}{c}(\hat{a}) = \frac{b'}{b} \cdot \frac{l'_\epsilon}{l_\epsilon} \cdot e^{-r\hat{a}} = 1,$

$$(21) \quad \log \frac{b'}{b} = \Delta r \hat{a} - \log \frac{l'_\epsilon}{l_\epsilon}$$

When the difference in  $r$  arises wholly from a fertility difference,  $\log \frac{l'_\epsilon}{l_\epsilon} = 0$ , and  $\log \frac{b'}{b}$  is equal to  $\Delta r \hat{a}$ . If  $\hat{a} = T$ , the ratio of the birth rates is the same as the ratio of fertilities. On the other hand, if infant mortalities differ and fertility rates are unchanged, the two stable populations will have the same birth rate when  $\hat{a} = T$ .

The combined effect of mortality improvements represented by curves A and B and Figure 5 are:

(1) To raise the growth rate by the sum of the slope of A, and the rise that would be caused by an increase in fertility equivalent to the extra improvement in mortality at early ages.

(2) To change the age distribution in a way nearly equivalent (for all but the very youngest ages) to a rise in fertility which would yield the same increased flow of 5-year olds.

(iii) *Curve C in Figure 5.* Finally, we turn to the effects of curve C in Figure 5—the extra improvement in  $\log \frac{l'_a}{l_a}$  in the older ages.

The expression for  $\frac{c'}{c}(a)$  for this case becomes simply:

$$(22) \quad \frac{c'}{c}(a) = \frac{b'}{b} \cdot \frac{l'_a}{l_a}$$

because  $\Delta r = 0$ . The intrinsic rate of increase is unaffected by changes in older age mortality since it is wholly determined by fertility rates and by mortality rates within and before the childbearing ages. Moreover,  $b'$  differs from  $b$  only because the greater proportion of older persons affects the relation of births to the total population: the ratio of births to the population under 50 is the same in both age distributions. The ratio  $\frac{b'}{b}$  may be estimated by first noting that

$$(23) \quad \int_0^{\omega} \left( \frac{c'(a)}{c(a)} - 1 \right) c(a) da = 0$$

or that

$$\int_0^{\omega} \left( \frac{b'}{b} \frac{l'_a}{l_a} - 1 \right) c(a) da = 0$$

Since

$$\left( \frac{b'}{b} \frac{l'_a}{l_a} - 1 \right) \cong \log \frac{b'}{b} + \log \frac{l'_a}{l_a},$$

$$(24) \quad \int_0^{\omega} \left( \log \frac{l'_a}{l_a} \right) c(a) da \cong \log \frac{b'}{b} \int_0^{\omega} c(a) da$$

but from 0 to  $a_0$ ,  $\log \frac{l'_a}{l_a} = 0$ , and  $\int_0^\omega c(a) da = 1$ . Hence

$$(25) \quad \log \frac{b}{b'} \cong \int_0^\omega \log \frac{l'_a}{l_a} c(a) da.$$

We may make a final approximation. There exists an  $\bar{a} > a_0$ , such that

$$\int_{a_0}^\omega \log \frac{l'_a}{l_a} c(a) da = \log \frac{l'_{\bar{a}}}{l_{\bar{a}}} \int_{a_0}^\omega c(a) da.$$

If  $\log \frac{l'_a}{l_a}$  rose linearly, and  $c(a)$  declined linearly, from  $a_0$  to  $\omega$ ,  $\bar{a}$  would lie one-third of the way along the interval from  $a_0$  to  $\omega$ . Thus when only a negligible fraction survive above age 90, the assumption of linearity in  $\log \frac{l'_a}{l_a}$  and in  $c(a)$  above  $a_0$  leads to the approximation:

$$(26) \quad \log \frac{b}{b'} \cong \left( \log \frac{l'_{\bar{a}}}{l_{\bar{a}}} \right) c(a_0 +)$$

where  $\bar{a} = \frac{2}{3} a_0 + \frac{1}{3} (\omega)$ .

In other words, the per cent difference in the birth rate is proportional to the per cent difference in  $l_a$  for a centroidal age above  $a_0$ , and to the fraction above  $a_0$  in the original age distribution.

The rather intricate reasoning in the discussion to this point may have obscured some of the more important results that the analysis implies. We will try to illuminate the more interesting conclusions—and incidentally show how the analysis works out in numerical form—by considering an empirical example.

The example is based on Swedish vital rates at the turn of the century and near the middle of the century. Specifically, it contrasts the stable age distribution accompanying Swedish fertility of 1896–1900 (gross reproduction rate 1.95) and mortality of 1891–1900 (expectation of life 53.6 years) with the age distribution implied by the fertility of 1950 and mortality of 1946–50 (gross reproduction rate 1.11, expectation of life 71.6

years). The most notable change (Fig. 6) is the increase in proportions at the older ages. Until recently, this sort of “aging” was commonly attributed to declines in both fertility and mortality.<sup>6</sup> However, it is clear from Figure 7 that improved mortality acting alone would have produced a *younger* distribution. The decline in fertility was not merely the principal force responsible for the “aging” population; it actually had to overcome an opposing force caused by the change in mortality.

These results are fully explained by our analysis. The difference between curves 1 and 3, and between curves 2 and 4 in Figure 7 is the result of the “pivoting” effect of a change in fertility—the point of pivoting being the average of the average ages. Note that the intersection of 2 and 4—which involves higher mortality—occurs at a slightly higher age, indicating that higher mortality produced a slightly greater average age.

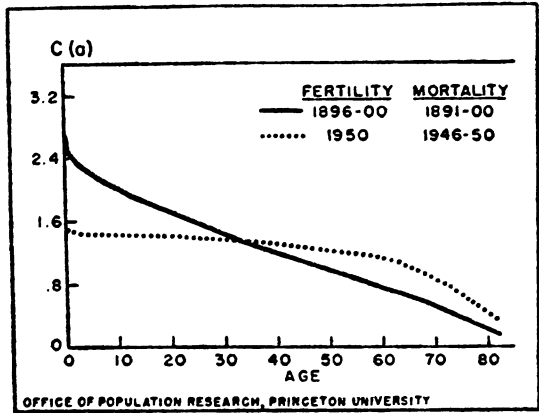


Figure 6. Stable age distributions for females in Sweden in 1896-1900 and in 1950.

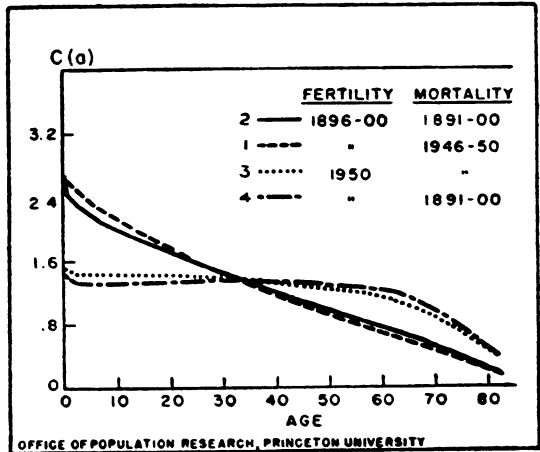


Figure 7. Stable age distributions associated with various combinations of fertility and mortality from Swedish experience around 1900 and 1950.

<sup>6</sup> Lorimer and Sauvy have both shown that the common belief that mortality reduction typically produces an older population is mistaken (Lorimer, 1951; Sauvy, 1954).



In accounting for the effect of mortality changes, we need to examine first the age pattern of mortality improvement. Figure 8 shows this pattern in the form of the natural logarithm of the ratio of survivors at each age in the two Swedish life tables.

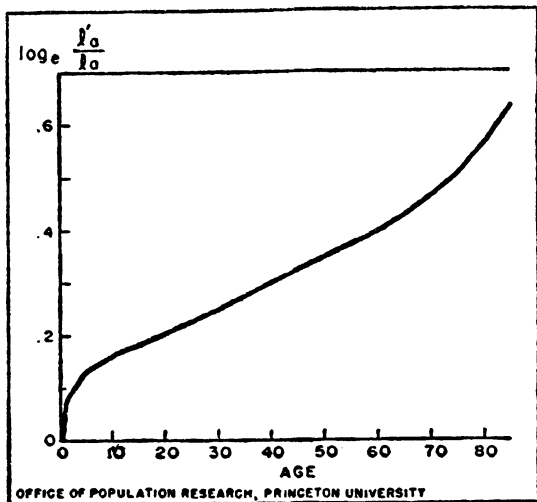


Figure 8. The natural logarithm of the ratio of the survivors to each age in the life tables for 1946-50 and 1891-1900 (Swedish females).

The pattern, it turns out, can be accurately represented by constituents A, B, and C (cf. Fig. 5). Thus the analysis presented in equations (18) through (26) is fully applicable.

With the aid of this analysis, we may note the following features of our example:

(1) The decline in fertility produced a larger effect than the decline in mortality on

the intrinsic rate of increase. Fertility acting alone would have reduced  $r$  by 17.7 per thousand, whereas the decline in mortality would have raised  $r$  by 8.2 per thousand.

(2) The increase in growth rate associated with the linear constituent A of changing mortality accounted for 4.7 per thousand of the change in  $r$  while the excess improvement in childhood mortality (B component) increased  $r$  by about 3.6 per thousand (Equation (19)).

Thus the decline in mortality, though it had nearly half as great an effect on growth as fertility, had a much smaller effect on the age distribution.

(3) The effect of the B component—the extra improvement in mortality in childhood—on the age distribution is equivalent to a 12 per cent rise in fertility, except for the effect on ages under 5. (See equation (20) and Fig. 9.) Note the effect of the extra improvement in mortality in the older ages (the C component) in causing a divergence at these ages from the age

pattern which would be caused by a 12 per cent rise in fertility.

(4) The age distribution effect of the same decline in childhood mortality is greater when fertility is held constant at a low rather than a high level. (See Fig. 10.) This result is understandable when one recognizes that the improvement in childhood mortality causes the age distribution to pivot approximately on the average age. With low fertility and a relatively great average age, this pivoting action raises the proportion of women at the most fertile ages, and the increased survival in early childhood is reinforced by a rise in the number of mothers. (See equations (20) and (21).)

(5) In spite of the C component (the extra mortality improvement in the older ages), the fraction at these ages is diminished, not increased, by reduced mortality. This result arises because the B component is so large, while the C component is only moderate.

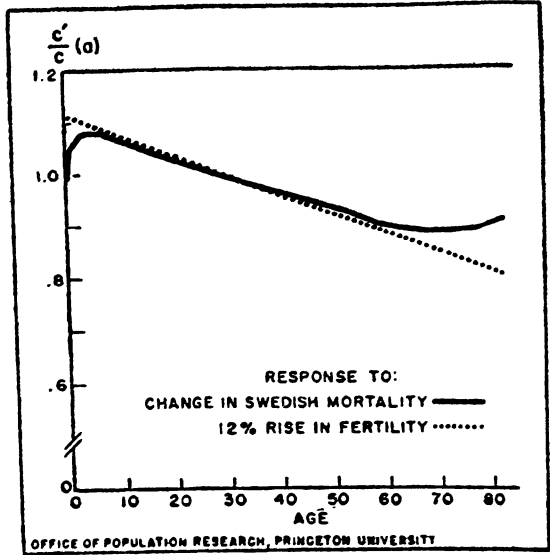


Figure 9.  $\frac{c'(a)}{c(a)}$  in response to an improvement in mortality, and in response to a nearly equivalent rise in fertility.

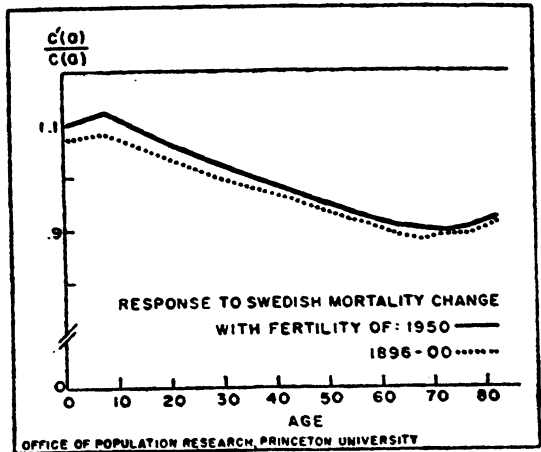


Figure 10.  $\frac{c'(a)}{c(a)}$  for various combinations of Swedish vital rates.

Some of the results illustrated by the Swedish example are universal—such as the nature of the effect of declining fertility. However, the results of changing mortality depend (we repeat) on the age pattern of the change; and of course the results in the example can be taken as typical only of instances with a similar pattern.

To get an idea of the prevalence of this pattern of changing mortality, we have surveyed a large number of pairs of life tables,<sup>7</sup> each pair contrasting the experience of some area at two different dates. Some of the results of this survey are presented below. The pairs of life tables are collected in groups on the basis of time interval spanned, and on the basis of rough similarity of mortality level during the same interval.

Group I consists of pairs of life tables for 11 European areas during the last half of the nineteenth century (life tables for Belgium, Berlin, Breslau, England and Wales, France, Germany, Italy, Netherlands, Prussia, Sweden, and Switzerland). Group II consists of tables for 15 countries with a relatively high expectation of life in the interval 1900–1941 (Australia, Canada, Denmark, England and Wales, France, Germany, Iceland, Netherlands, Norway, Sweden, Union of South Africa, and the United States). Group III is made up of life table pairs for 9 countries with relatively high mortality in the twentieth century (British Guiana, Chile, Ceylon, Jamaica, Japan, Mexico, Portugal, Taiwan, and Trinidad and Tobago). Group IV consists of life tables for 9 countries with a high expectation of life and a substantial improvement in mortality during the interval covered—from the 1930's to the present.

We will first summarize the linearity of  $\log \frac{l'_a}{l_a}$  from ages 5 to 50 for the life table pairs in these groups.

It is notable that 39 out of 44 life table pairs have a  $\log \frac{l'_a}{l_a}$

<sup>7</sup> This survey was made easy by access to the life tables collected by Professor George J. Stolnitz of the Office of Population Research—a collection including substantially all of the national life tables ever published, and a large number of life tables for non-national areas as well.

GROUP	DEVIATION <.01	DEVIATION .01 to .02	DEVIATION .02 to .03	DEVIATION .03 to .04
I. Europe, 1850-1900	6	5	0	0
II. Low Mortality, 1900-1940	13	2	0	0
III. High Mortality, 1900-	3	1	2	3
IV. Low Mortality, 1935-	9	0	0	0

Table 1. Number of life table pairs with given maximum deviation from straight line of best fit,  $\log \frac{l'_a}{l_a}$  from age 5 to age 50, by groups.

function linear within 2 per cent from 5 to 50. The maximum departure from linearity shows the maximum error one would make in estimating the altered stable age distribution on the assumption that the change *was* linear. In nearly 90 per cent of the life table pairs this error would be less than 2 per cent. (Less than 2 per cent of the size of the group itself. Thus if the group 20-24 actually constitutes 10 per cent of the population after the improvement in mortality, it would be estimated within the interval 9.8 to 10.2 per cent.)

The significance of departures from linearity can be better appreciated when we observe what linearity in  $\log \frac{l'_a}{l_a}$  implies about changes in the probability of surviving. If  $\log \frac{l'_a}{l_a}$  is linear over an interval, the probability of surviving changes by the same proportion at all of the ages the interval covers. Thus if  $\log \frac{l'_a}{l_a}$  is linear from 5 to 50, the probability of surviving from age 5 to age 6 is increased by the same factor as the probability of surviving from 30 to 31, from 45 to 46, etc. If the probability of surviving a year is close to unity at some ages—say .99 or more—there is *room* for only a slight proportional increase. It turns out, in fact, in all three cases listed in Table 1 of a departure of more than .03 from linearity, that if the average improvement from 5 to 50 in the probability of surviving had applied at all ages, some mortality rates would have become negative. In other words, the most prominently nonlinear mor-

tality improvements *could not possibly* have been linear—the improvement was too large. Even in these cases, however, the use of a linear relationship as an approximation to the actual change produces only a slightly inexact estimate of the resulting stable age distribution.

Next we will consider the prevalence of B and C components in the mortality improvement patterns. In all forty-four life table pairs the probability of surviving to age 5 increases in greater proportion than the change in any other 5-year probability under age 50. In other words, the existence of a noticeable positive B component is universal for this selection of mortality changes. However, among group IV (most recent experience of low mortality areas) the B component is less pronounced. The C component (excess improvement in the older ages) is pronounced in all instances except 5 in group I, 4 in group II, 2 in III, and 1 in group IV. The proportional increase in the probability of surviving above age 50 typically exceeds the increase for 5 to 50 by the widest margin in group IV. In short, the C component clearly exists in thirty-two of forty-four pairs of life tables. In the recent experience of advanced countries, it is becoming more pronounced relative to the other components.

The last part of our survey will examine the effect of changes in mortality on the median age of the stable age distribution. The results are shown in Table 2.

Table 2 was prepared by applying the analysis presented in Figures 3 and 4 to  $\log \frac{l'_a}{l_a}$  for all life table pairs. The reason for

Table 2. Effect of reduced mortality on the median age of the stable population.

GROUP	MEDIAN AGE LOWERED WHETHER FERTILITY IS HIGH OR LOW	MEDIAN AGE LOWERED ONLY IF FERTILITY IS HIGH	MEDIAN AGE RAISED
I	10	1	0
II	15	0	0
III	8	1	0
IV	2	3	4

differentiating the first two columns can be seen from Figure 4.

When  $\log \frac{l'_a}{l_a}$  rises steeply enough in the older ages to produce an increase in the fraction above some older age, say 67 years, this effect will *raise* the median age if the original fraction over 67 is high; otherwise the rise in the proportion aged will be swamped by the increase in very young. But whether there is or is not a large fraction over 67 in the initial stable distribution depends on whether fertility is high or low. Life table pairs assigned to column 1 of Table 2 either show a decrease at all ages above the original median or show a decrease at all ages up to a very advanced age. In these instances the median age is lowered without regard to the fertility level. In the second column, the  $\frac{c'(a)}{c(a)}$  fractions associated with the life table pairs have a substantial positive area above age 60 or so, and a substantial positive area below age 20. Which of these predominates in affecting the median age depends on the fertility level. The four life table pairs in column 3 combine pronounced old age improvement with a relatively meager improvement in childhood mortality.

Our survey shows that our Swedish example is typical of a wide range of vital rate changes in the following ways:

(1) The effect of declining fertility is universally to lower the growth rate, increase the fraction at ages under the average age, and decrease the fraction at higher ages.

(2) The majority of the life table pairs examined resembled the Swedish mortality change in the approximate linearity of the A component and in having positive B and C components.

(3) The B component tends to be less pronounced and the C component more pronounced in the recent experience of low mortality areas.

(4) In all but a small fraction of instances examined, the effect of mortality improvement would be to lower the median age of the stable age distribution.

Since mortality rates up to age 50 have reached very low

levels in the areas of lowest mortality today, there is very little room for improvements having the effect of making the population younger. For example, if all deaths under age 5 were eliminated in Sweden, and no other mortality changes occurred (this is the *maximum* B component), the effect would be equivalent only to a 2.5 per cent increase in fertility. In other words, further substantial improvements in mortality in the regions which today have the highest expectations of life will have to occur in ages above 50, where the age distribution effect is in one direction only—that of producing an older population.

#### TRANSITORY CHANGES IN AGE DISTRIBUTIONS ARISING FROM CHANGES IN MORTALITY AND FERTILITY RATES

The stable age distribution ultimately associated with a schedule of vital rates clearly contrasts the effects of different changes in fertility and mortality. However, the stable age distribution is sometimes approached only after a long interval—perhaps 60 years or more—of approximately constant rates. When fertility changes are substantial, or when sharp decreases in infant and child mortality occur, the transitory age distribution may depart markedly from the stable form. In view of the increasingly uncertain course of human affairs as one looks further into the future, these “transitory” age distributions often have more practical interest than the remote stable distribution.

The conventional method for calculating the population resulting from a particular course of vital rates is component projection. This method is flexible enough to handle any course whatever of age specific rates, and any initial age distribution. But it is not always clear in a component projection whether particular features of a projected age distribution are the result of birth rate changes, death rate changes, or of the character of the initial age distribution. The role of the various factors can be clarified, however, by considering projections of certain kinds of initial populations with certain simple changes in vital rates. Specifically, we will make the following simplifying assumptions:

- (1) The initial age distribution is the stable age distribution implicit in the initial vital rates.
- (2) The changes, if any, in fertility affect all age-specific rates in the same proportion.
- (3) Mortality changes, if any, conform to the typical pattern of life table changes described earlier.

The number at each age in the initial population will be designated as  $n(a, 0)$ ; the number there would be at time  $t$  if *vital rates remained unchanged* will be designated  $n(a, t)$ ; while the number at time  $t$  *with changing vital rates* will be designated  $n'(a, t)$ . Our analysis will obtain expressions for the ratio of the population at each age with changing vital rates to the population that would exist with no changes—expressions for  $\frac{n'(a, t)}{n(a, t)}$ , which we will designate  $f(a, t)$ . Fertility and mortality changes will be considered separately.

*The Effects of Fertility Changes.* The fertility of all age groups will be assumed to change in the same proportion. Thus  $\frac{m(a, t)}{m(a, 0)} = g(t)$ —all age-specific fertility rates have the same time pattern. Under these circumstances:

$$(1) f(0, t) = \frac{B'(t)}{B(t)} = g(t) \text{ for } t \leq a_1$$

where  $B'$  is the number of births with changing fertility,  $B$  is the number of births that would have occurred with initial fertility;  $g(t) = \frac{m(a, t)}{m(a, 0)}$  is the ratio of age-specific fertility at time  $t$  to that at time 0; and  $a_1$  is the earliest age at which a significant rate of childbearing occurs.

For the first  $a_1$  years, in other words, births will simply change in the same proportion as the change in fertility. When the interval following the initial change equals and then exceeds the earliest age of childbearing, the number of births begins to reflect two factors—the current fertility level and the altered number of mothers.



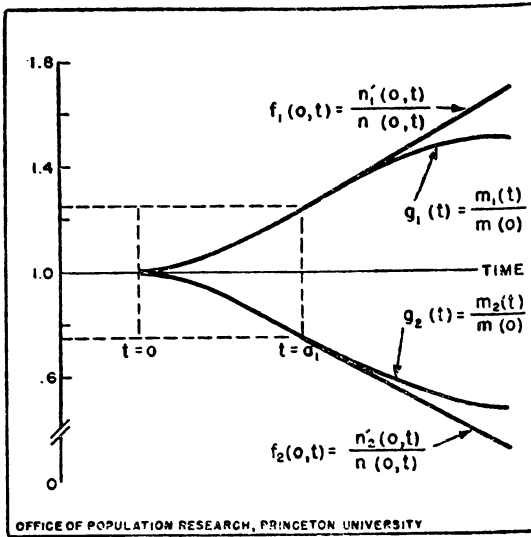


Figure 11. The effect on the number of births of increasing fertility [ $g_1(t)$ ] and declining fertility [ $g_2(t)$ ].

Similar reasoning applies to declining fertility, as shown by  $g_2(t)$  and  $f_2(0, t)$ .  $f(0, t)$  may be expressed in the following functional relation:<sup>8</sup>

<sup>8</sup> Equation (2) can be derived as follows:

$$(a) f(0, t) = \frac{\int_0^{\omega} n'(a, t)g(t)m(a) da}{\int_0^{\omega} n(a, t)m(a) da}$$

since the numerator gives total births from  $n'(a, t)$  mothers and age-specific fertility  $g(t)m(a)$ , and the denominator gives total births when fertility remains constant. (a) can be rewritten as

$$(b) f(0, t) = g(t) \frac{\int_0^{\omega} f(a, t)n(a, t)m(a) da}{\int_0^{\omega} n(a, t)m(a) da}$$

and if it is noted that  $f(a, t) = 1$  for  $a > t$ , (b) becomes

$$(c) f(0, t) = g(t) \left\{ \frac{\int_0^t \{f(a, t) - 1\}n(a, t)m(a) da}{\int_0^{\omega} n(a, t)m(a) da} + 1 \right\}$$

Finally, if we note that  $f(a, t) = f(0, t-a)$  because of constant mortality, and in turn that  $f(0, t-a) = g(t-a)$  for  $t-a < a_1$ , and lastly that  $m(a) = 0$  for  $a < a_1$ , (2) follows from (c).

Figure 11 illustrates the relation between  $g(t)$  (the proportionate change in fertility) and  $f(0, t)$  (the proportionate change in births). When fertility is rising ( $g_1(t)$ ), the consequent rise in births follows fertility precisely—until the number of mothers is affected. From then on the proportionate rise in births exceeds the proportion by which fertility has risen.

$$(2) f(0, t) = g(t) \left\{ \frac{\int_{a_1}^t \{g(t-a) - 1\} c(a, t) m(a) da}{\int_{a_1}^{a_2} c(a, t) m(a) da} + 1 \right\}$$

provided  $t \leq 2a_1$ .

So far we have made no use of the assumption that the initial population had a stable age distribution. In fact (2) holds generally, with  $c(a, t)$  designating the fraction of the population that would have been at age  $a$  at time  $t$  if fertility had remained constant. The assumption of an initial stable age distribution simply enables us to rewrite (2) replacing  $c(a, t)$  by  $c(a)$ . Since we have assumed mortality constant,  $f(a, t) = f(0, t - a)$ .  $f(a', t)$  may be written:

$$(3) f(a', t) = g(t - a') \left\{ \frac{\int_{a_1}^{t-a'} \{g(t-a'-a) - 1\} c(a, t - a') m(a) da}{\int_{a_1}^{a_2} c(a, t - a') m(a) da} + 1 \right\}$$

Again, the assumption that the initial age distribution is stable will permit us to replace  $c(a, t - a')$  by  $c(a)$ . This formidable-looking expression has the following meaning. At a given time  $t$ , all cohorts born since time 0 have been affected by changing fertility. If fertility has been rising monotonically, the cohorts have been progressively enlarged. But because of the assumed constant mortality, the size of a cohort relative to what its size would have been is set at birth. Thus the cohort born  $t - 1$  years ago (the cohort with  $a = t - 1$ ) was determined by the fertility change in the first year after the change in fertility began. If  $f(0, t)$  is plotted as in Figure 12, from time 0 to time  $t'$ , then  $f(a, t')$  will be unity for  $a \geq t'$ ; for  $a < t'$ ,  $f(a, t')$  can be obtained by plotting  $f(0, t)$  from right to left beginning at  $a = t'$ .

Equation (3) will hold precisely provided  $t \leq 2a_1$ . As  $t$  exceeds  $2a_1$ ,  $f(0, t)$  will begin to be affected by second generation births. However, the number of second generation births will

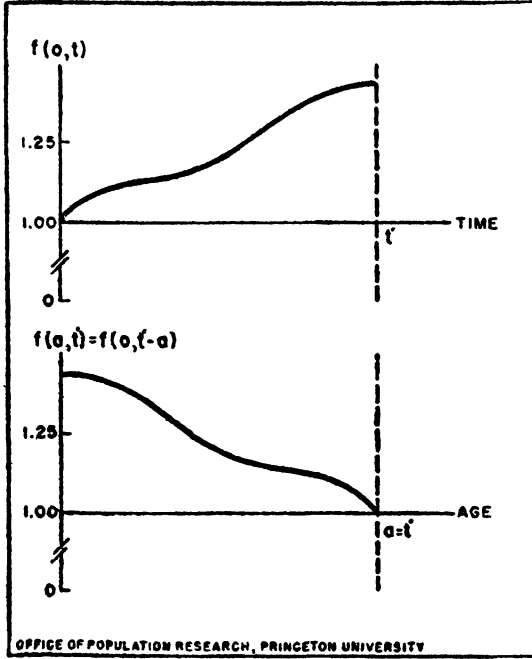


Figure 12. The relation between changing number of births [ $f(0, t)$ ] and changes in the numbers at each age [ $f(a, t)$ ], with changing fertility.

each age, through the relation  $n(a, t') = n(a, 0)e^{t'a}$

(b) Calculate  $f(a, t')$  for each  $a < t'$  by using equation (3). Note that  $f(a, t') = 1$  for  $a \geq t'$ .

(c) Calculate  $n'(a, t') = f(a, t') \cdot n(a, t')$ .

In most instances calculation of  $n'(a, t')$  at 5-year age intervals would be sufficient, with other ages estimated by linear interpolation.

*The Effects of Mortality Changes.* We will now analyze  $f(a, t)$  as age-specific mortality rates change. Here our assumptions will be:

(a) The initial age distribution is the stable distribution associated with the initial vital rates.

(b) Age-specific fertility rates remain constant.

(c) Age-specific death rates change so as to produce a pat-

be negligible for several years, and equation (3) will be a serviceable approximation for 35 or 40 years, assuming  $a_1 = 15$  years.

The following steps would be necessary to make a numerical projection of  $f(a, t')$ , for  $t' \leq 35$  or 40 years, with a population initially stable, and with fertility given by  $\frac{m(a, t)}{m(a, 0)} = g(t)$ :

(a) Calculate the population that there would have been at

tern of  $\log \frac{l_a(t)}{l_a(0)}$  similar to the pattern described earlier as typical. Thus  $\log \frac{l_a(t)}{l_a(0)}$  is the sum of three components;

- (i)  $\log \frac{l_a(t)}{l_a(0)} = u(t) \cdot a$  for all  $a$
- (ii)  $\log \frac{l_a(t)}{l_a(0)} = v(t)$  for  $a \geq 5$
- (iii)  $\log \frac{l_a(t)}{l_a(0)} = w(t) \cdot (a - a_0)$  for  $a > a_0$ ;  $a_0 \geq 50$

We will not give an analytical expression to the portion of the curve between ages 0 and 5, but will only assert that  $\log \frac{l_1(t)}{l_1(0)}$

is typically about half of  $\log \frac{l_5(t)}{l_5(0)}$ . Except for this interval, the

three time functions, (i), (ii), and (iii), determine a relationship  $\log \frac{l_a(t)}{l_a(0)}$  of the form shown by the heavy line in Figure 13.

The relation of  $l_a(t)$  to  $l_a(0)$  is specified by three parameters: the slope  $\alpha$  of the straight line portion drawn through the origin [ $\alpha = u(t)$ ]; the height  $\beta$  of  $\log \frac{l_5(t)}{l_5(0)}$  above  $5\alpha$  [ $\beta = v(t)$ ]; and

the extra slope  $\gamma$  ( $\alpha + \gamma =$  total slope) of  $\log \frac{l_a(t)}{l_a(0)}$  when  $a > a_0$ .

The basis for this representation of life table change is empirical; very many pairs of life tables from actual experience fit this pattern closely. (*See* discussion of life tables, p. 97.)

We will use the following procedure: (a) we will determine  $f_1(a, t) = \frac{n_1(a, t)}{n(a, t)}$  where  $n_1$  is the number at age  $a$  that there would be in response to the linear relation  $\log \frac{l_a(t)}{l_a(0)} = \alpha a$  [ $\alpha = u(t)$ ] acting alone; (b) we will next determine  $f_2(a, t) = \frac{n_2(a, t)}{n_1(a, t)}$ , where  $n_2$  is the number resulting from a rise of mag-

nitude  $\beta$  [ $\beta = v(t)$ ] above  $5 \cdot \alpha$  of  $\log \frac{l_5(t)}{l_5(0)}$ ; and (c) we will finally determine  $f_3(a, t) = \frac{n_3(a, t)}{n_2(a, t)}$ , where  $n_3$  is the number at

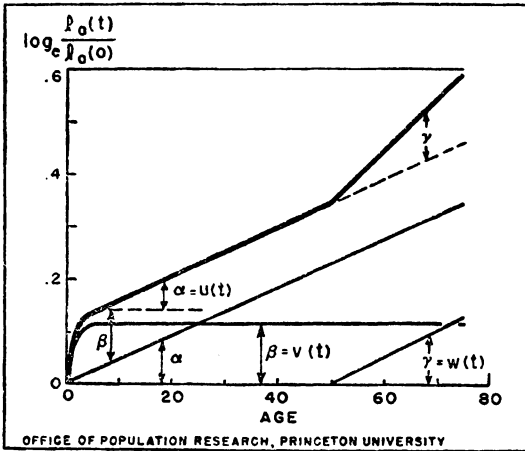


Figure 13. The components of  $\log_e \frac{l_a(t)}{l_a(0)}$ .

age  $a$  resulting from the extra slope  $\gamma$  above age  $a_0$ . Then the effect of the complete change in mortality rates will be  $(f_1 \cdot f_2 \cdot f_3) = f(a, t)$ .

(a) The effect of  $\alpha a$ , the linear component of mortality change.

If the only change in mortality from year 0 to year  $t$  were of the

form  $\log \frac{l_a(t)}{l_a(0)} = u(t)a$ ,

the per cent age distribution in later years would be the same that would have prevailed with mortality unchanged. We will show this for year 1. The life table at the end of six months is related to the original table by an exponential factor:

$$l_a(1/2) = l_a(0)e^{u(1/2)a}$$

if according to the former mortality a certain fraction  $S(a)$  would have survived from age  $a$  to  $a + 1$ , according to the changed mortality, a fraction equal to  $S(a)e^{u(1/2)}$  will survive. The fraction surviving at every age is multiplied by the same factor. Hence there would be  $e^{u(1/2)}$  times as many persons at every age (including age zero, since parents are increased by the common factor, and fertility is unchanged), and thus the *proportion* at every age is left precisely as if mortality had remained constant. However, the *number* at every age is  $u(1/2)$  per cent greater than it would have been.

After  $t$  years have passed,  $f_1(a, t)$  will be given by:

$$(4) f_1(a, t) \cong e^{[u(1/2) + u(3/2) + \dots + u(t-1/2)]}$$

or more exactly,

$$(5) f_1(a, t) = e^{\int_0^t u(t) dt}$$

If the population at time 0 were stable with an intrinsic rate of increase  $r$ ,  $n(a, t)$  equal  $n(a, 0)e^{rt}$  and the number resulting from the linear component of changing mortality would be

$$(6) n_1(a, t) = \left( e^{rt + \int_0^t u(t) dt} \right) n(a, 0)$$

(b) The effect of  $\beta [v(t)]$ —the excess at age 5 of  $\log \frac{l_a(t)}{l_1(0)}$  over  $5 \cdot \alpha$ .

The effect of a life table change of the form sketched in Figure 14 is similar to that of a rise in fertility. Consider a cohort reaching age 5 at time  $t$  ( $t \leq 15$ ). The cohort will be larger than it would have been, because of decreases in mortality. If the entire improvement in mortality under age 5 were concentrated in the first day of life, we could determine  $f_2(5, t)$  by finding out the increase in survivorship of the first day of life that had occurred five years before  $t$ —in short, we could assert:

$$(7) f_2(5, t) = e^{v(t-5)}$$

On the other hand, if the entire improvement in mortality under 5 were concentrated in the last day before the fifth birthday, we could assert:

$$(8) f_2(5, t) = e^{v(t)}$$

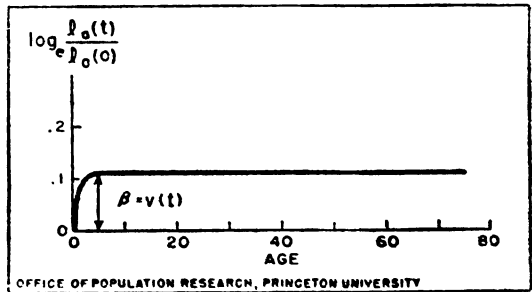


Figure 14. The excess improvement in childhood mortality  $\beta = v(t)$ .

But if the improvement in mortality were spread out—as it usually is—from the first to the last day of the five-year span, then we can write

$$(9) f_2(5, t) \cong e^{v(t-\bar{a})}$$

where  $\bar{a}$  is a time lag somewhere intermediate between 0 and 5. Since, indeed, the improvement is typically more pronounced in the earlier ages of the interval, a lag of 3 or 3 1/2 years is a fair approximation to  $\bar{a}$  in (9).

Once a cohort has reached age 5, changes in mortality below that age will not affect its size. The analogy between  $f_2(a, t)$  and the change in numbers occasioned by a rise in fertility is now apparent. There are two factors influencing  $f_2(a, t)$  for  $a \geq 5$ —the first is any change in the size of the cohort at birth, and the second is the changed probability it experienced in surviving to age 5. But  $f_2(0, t)$ —the proportionate increase in a birth cohort—is determined by changes in the number of mothers, just as the number of mothers helped determine the birth cohort when fertility was changing. (Cf. equation (2) above, p. 104.)

The end result is that if we designate  $e^{r(t)} = h(t)$ , we obtain an expression for  $f_2(a, t)$  very much like that obtained for  $f(a, t)$  when fertility changes were analyzed.

$$(10) \quad f_2(a', t) = h(t - a' + 2) \left\{ \frac{\int_0^{t-a'} [h(t - a + 2) - 1] c(a) m(a) da}{\int_{a_1}^{a_2} c(a) m(a) da} + 1 \right\}$$

provided  $t$  is less than 35 or 40 years.

(c) The effect of  $\gamma [w(t)]$ , the excess slope of  $\log \frac{l_a(t)}{l_a(0)}$  for  $a > a_0$ .

A straight line  $\log \frac{l_a(t)}{l_a(0)}$  rising at an angle  $\gamma$  beginning at  $a = a_0$  is equivalent to an increase by a factor of  $e^\gamma$  in the probability of surviving for one year at every age above  $a_0$ . As each cohort passes age  $a_0$ , it acquires an incremental growth rate  $w(t)$ . Thus at time  $t'$ , any cohort at age  $a > a_0$  will have gained in the following proportion:

$$(11) f_3(a, t') = e^{\int_{t'-(a-a_0)}^{t'} w(t) dt} \quad \text{(where the minimum lower limit of the integral is zero)}$$

The combined effect of the three components of mortality change is:  $f(a, t) = (f_1)(f_2)(f_3)(a, t)$

$$(12) f(a', t') = \left( e^{\int_0^{t'} u(t) dt} \right) h(t' - a' + 2) \left\{ \frac{\int_0^{t'-a'} [h(t' - a + 2) - 1] c(a) m(a) da}{\int_{a_1}^{a_2} c(a) m(a) da} + 1 \right\} e^{\int_{t'-a'+a_0}^{t'} w(t) dt} \quad \text{where } h(t) = e^{v(t)}$$

The effect of a simultaneous change in fertility, described by  $\frac{m(a, t)}{m(a, 0)} = g(t)$ , and in mortality, described by  $\log \frac{l_a(t)}{l_a(0)} = u(t) \cdot a + v(t) + w(t)(a - a_0)$  is obtained by using the product of  $g(t)$  for fertility and  $h(t+2)$  for mortality in place of simply  $h(t+2)$  in (12). It is interesting to note that if  $\frac{1}{g(t)} = h(t+2)$ , a decline in fertility would be precisely offset by the decline in childhood mortality (in excess of the average mortality decline, so to speak). Under these circumstances, the only age distribution effects of changed vital rates would be a slight decline in the fraction under 5 because of the fact that child mortality improvement is spread over the interval 0-5 rather than concentrated in the first few days of life, and possibly a rise in the fraction at older ages, due to the above-average improvements in mortality at these ages.

The transitory age distribution changes—like the changes in the stable distribution—are easier to picture with the help of an example. Our examples will by means of graphs show the effects of very sudden “step-function” changes in fertility and mortality.

Assume (as a first example) that there is a 10 per cent rise



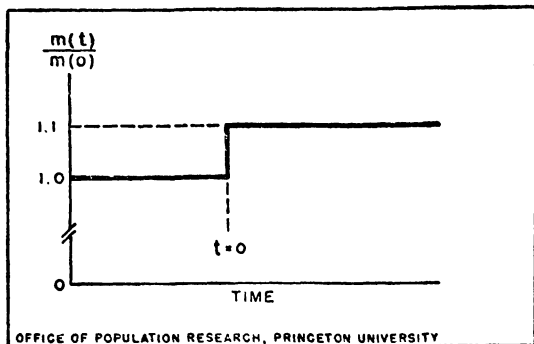


Figure 15. Ten per cent "step-function" rise in fertility.

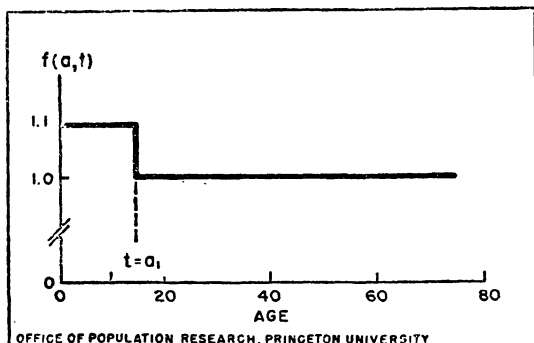


Figure 16.  $f(a, t) = \frac{n'(a, t)}{n(a, t)}$ , in response to a ten per cent "step-function" rise in fertility  $t$  years after the rise when  $t = a_1$ .

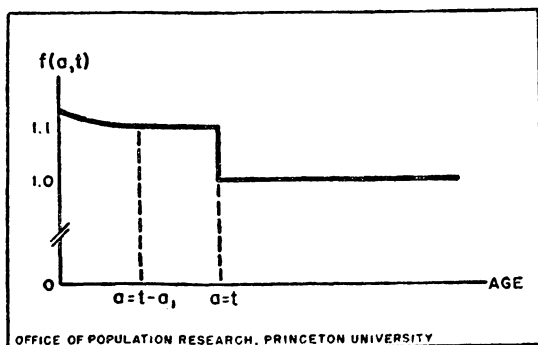


Figure 17.  $f(a, t) = \frac{n'(a, t)}{n(a, t)}$ , in response to a ten per cent "step-function" rise in fertility  $t$  years after the rise when  $t > a_1$ .

in fertility at  $t = 0$ , and that fertility remains at the new high levels from then on. (Fig. 15.)

Births will rise by 10 per cent, and remain 10 per cent above the formerly expected value until the number of mothers is affected.

When  $t = a_1$ ,  $\frac{n'(a, t)}{n(a, t)}$  will have the appearance shown in Figure 16. But after  $t > a_1$ ,  $n'(a, t)$  for the youngest ages will begin to reflect an increased number of mothers (Fig. 17).

Later the increased daughters themselves swell the number of mothers, so that by the time  $t$  approaches  $\omega$ , the number of births will have been increased by perhaps 35 per cent, as in Fig. 18.

If we consider  $\frac{c'(a, t)}{c(a, t)}$  (proportionate age distributions instead of numbers at each age), the increase in the proportion

formed by the younger age groups must be offset by a decrease in the proportion in the older ages. In short, the proportionate parts of the new age distribution must still add up to one. Thus  $\frac{c'(a, t)}{c(a, t)}$  cannot lie wholly above or on

unity as does  $\frac{n'(a, t)}{n(a, t)}$  in Figure 18. In fact, when  $t = \omega$ ,  $\frac{c'(a, t)}{c(a, t)}$  very closely resembles the relation between two stable age distributions with a 10 per cent fertility difference.

In our second example, we will consider mortality improvements which take the form of a step function of time. If  $u(t)$ —the linear component of mortality change—rises from 0 to .01 at  $t = 0$  and remains at .01 from then on, the effect is simply to raise the growth rate

by 1 per cent. In other words,  $\frac{n'(a, t)}{n(a, t)} = e^{.01t}$ . (Fig. 19.)

If  $v(t)$ —the excess improvement in mortality under 5—increases so as to produce a sudden 10 per cent improvement in the probability of surviving to age 5, the effect is very similar

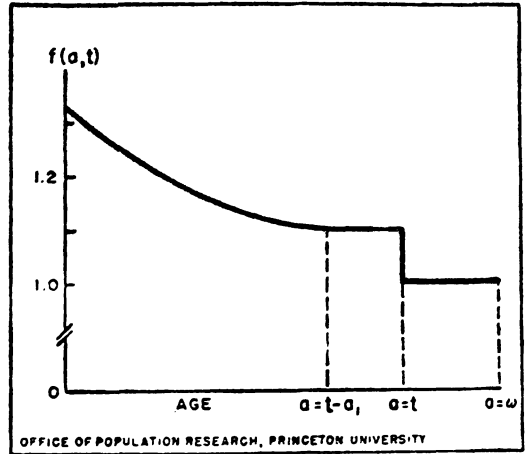


Figure 18.  $f(a, t) = \frac{n'(a, t)}{n(a, t)}$ , in response to a ten per cent "step-function" rise in fertility  $t$  years after the rise as  $t$  approaches  $\omega$ .

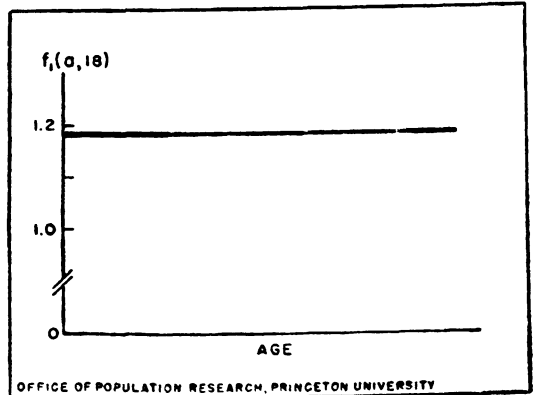


Figure 19.  $f_1(a, t)$  in response to a one per cent increase in the probability of surviving at all ages 18 years after the change.

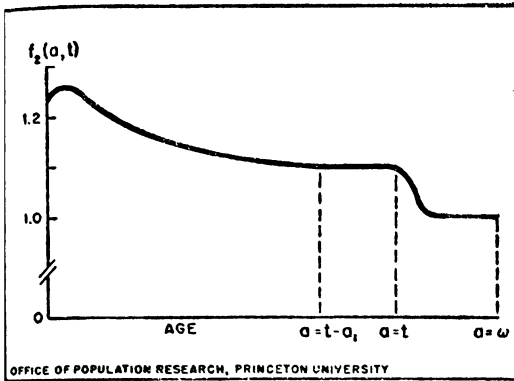


Figure 20.  $f_2(a, t)$  in response to a ten per cent increase in the probability of surviving to age 5,  $t$  years after the increase.

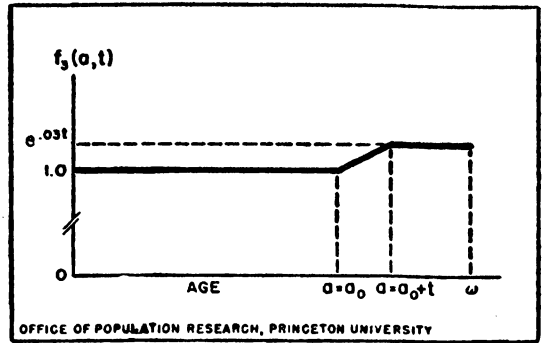


Figure 21.  $f_3(a, t)$  in response to a sudden increase of three per cent in the probability of surviving at ages above  $a_0$ .

to that portrayed in Figures 16 through 18. The principal difference is that the wave-front is no longer vertical. (Fig. 20.)

Lastly, if  $\omega(t)$ —the excess improvement in mortality in the older ages—increases so as to produce a sudden increase in the steepness of  $\log \frac{l(a, t)}{l(a, 0)}$  for  $a > a_0$ , the effect is, of course, to cause

a rise in the numbers above  $a_0$ . If the slope of  $\log \frac{l(a, t)}{l(a, 0)}$  is suddenly increased by .03, the effect  $t$  years later is shown in Figure 21.

The time pattern of changing rates used in these examples represents the sharpest possible sort of transient change. The examples show, in qualitative terms, that transitory effects on age distributions are in the same general direction as the eventual effects on the stable age distribution. A rise in fertility produces an increase in the proportion in the younger age groups at the expense of the older; a proportionate increase at all ages in the probability of surviving affects only the growth rate; an extra increase in survivorship at the youngest ages has an effect much like a rise in fertility; an extra increase in survivorship at the older ages tends to raise the fraction at these ages.

These conclusions are (we repeat) qualitative and inexact.

The *exact* effect on a particular age depends on too many factors to be described simply. However, one can say with every confidence of being correct that a lower course of fertility produces an older population than would a higher course, all other factors being the same; and with fair confidence that most mortality improvements in the past have produced a younger population than would have resulted from unchanged mortality, other factors the same. It seems clear, moreover, that further improvements in mortality in those areas currently having the lowest mortality risks will tend to produce an older population.

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